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Actions of unitary tensor categories (UTC) on C^*-algebras

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Outline:

- Introduction: interplay between subfactors and UTCs.
  - von Neumann algebras v.s. C^*-algebras
- UTC-actions on C^*-algebras:
  - Constructing UTC-actions
  - Perspectives

A UTC is a tuple $(\mathcal{C}, \otimes, \alpha, \mathbb{1})$, where

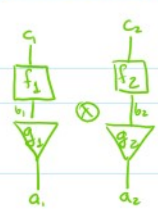
- $\mathcal{C}$ is a countably semisimple $\mathbb{C}$-linear category,
- the tensor product
$$-\otimes- : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$$
is a bilinear functor,
- α is the associator, and
- $\mathbb{1}$ is the simple monoidal unit,

Satisfying certain coherence axioms like the pentagon:

$$\begin{tikzcd}
 & ((a\otimes b)\otimes c)\otimes d \arrow[dl, "\alpha_{a,b,c}"'] \arrow[dr, "\alpha_{a\otimes b,c,d}"] & \\
(a\otimes(b\otimes c))\otimes d \arrow[d, "\alpha_{a,b\otimes c,d}"'] & \circlearrowright & (a\otimes b)\otimes(c\otimes d) \arrow[d, "\alpha_{a,b,c\otimes d}"] \\
a\otimes((b\otimes c)\otimes d) \arrow[rr, "\alpha_{b,c,d}"'] & & a\otimes(b\otimes(c\otimes d))
\end{tikzcd}.$$

Properties and structure:

- Every endomorphism algebra $\mathcal{C}(c)$ is a finite dimensional C^*-algebra.
- Every object $c\in\mathcal{C}$ has a unitary dual $\bar{c}$ and predual $c_\vee$, $\overline{c_\vee} \cong c$.
- Has a graphical calculus:

$(f_1\otimes f_2)\circ(g_1\otimes g_2) =$ 

$= (f_1\circ g_1)\otimes(f_2\circ g_2)$

$$\in \mathcal{C}(a_1\otimes a_2 \longrightarrow c_1\otimes c_2)$$

Introduction: A subfactor is a unital inclusion of $\underbrace{\text{simple von Neumann algebras}}_{\text{(factors)}}$

$$A \subset B.$$

Eg: • $Mat_2(\mathbb{C}) \subset Mat_{2^2}(\mathbb{C})$

•• $\underset{\text{disc. gp.}}{\Gamma} \underset{\text{free}}{\overset{\alpha}{\curvearrowright}} \underset{\text{The hyperfinite } II_1\text{-factor}}{R}$ 'the smallest infinite-dimensional factor' $\rightsquigarrow R^\Gamma \subset R \subset R \rtimes_\alpha \Gamma$

In [Jon83] Vaughan Jones introduced the index for subfactors, and showed it takes values in

$$\{4\cos^2\tfrac{\pi}{n}\}_{n\geq 3} \cup [4,\infty], \quad (!!)$$

and constructed $R \subset R$ for each allowed value. This development started the industry of subfactor classification by index.

Idea: Jones' Basic Construction

$$A \subset B \subset B_2,$$

where $[B_2 : B] = [B : A]$. (!)

Compare with Galois.

We study a subfactor through its standard invariant: Lattice of higher relative commutants:

$$\begin{array}{ccccc} A'\cap A & \subset A'\cap B & \subset A'\cap B_2 & \subset \cdots \\ & \cup & \cup & \\ & B'\cap B & \subset B'\cap B_2 & \subset \cdots \end{array}$$

Axiomatized as

- • Ocneanu's Paragroups
- •• Popa's λ-lattices / Commuting Squares
- ••• Jones' Planar Algebras
- ∴ Unitary Tensor Categories.

Reconstruction: Popa showed in [Pop95] that every λ-lattice arises as the standard invariant of some finite index subfactor.

revisit: Together with Shlyakhtenko in [PopShly03] showed it can be done using only

$$L\mathbb{F}_\infty \subset L\mathbb{F}_\infty.$$

key! Guionnet–Jones–Shlyakhtenko reproved Popa's reconstruction for subfactors using Planar Algebras and Free Probability [GJS10].

Eg: From $A \subset B \overset{GNS}{\rightsquigarrow} L^2B$ as A–B bimodule $\rightsquigarrow \langle L^2B, \oplus, \boxtimes, \bar{\cdot}\rangle$

$$\begin{array}{ccccccc} \underbrace{End_{A-A}(L^2A)}_{A'\cap A} & \subset & \underbrace{End_{A-B}(L^2B)}_{A'\cap B} & \subset & \underbrace{End_{A-A}(\overbrace{L^2B \boxtimes_A L^2B}^{L^2B_2})}_{A'\cap B_2} & \subset & \underbrace{End_{A-B}(\overbrace{L^2B \boxtimes_A L^2B \boxtimes_A L^2B}^{L^2B_3})}_{A'\cap B_3} \subset \cdots \\ & & \cup & & \cup & & \cup \\ & & \overbrace{End_{B-B}(L^2B)}^{B'\cap B} & \subset & \overbrace{End_{B-A}(L^2B)}^{B'\cap B_2} & \subset & \overbrace{End_{B-B}(\underbrace{L^2B \boxtimes_A L^2B}_{L^2B_2})}^{B'\cap B_3} \subset \cdots \end{array}$$

Abstractly, the standard invariant of $A \subset B$ corresponds to

Abstractly, the standard invariant of $A \subset B$ corresponds to a UTC $\mathcal{C}$, a chosen generator $x \in \mathcal{C}$,

and $\mathcal{C} \overset{\otimes}{\hookrightarrow} \mathrm{Bim}(A \oplus B)$,

recovering

$$\begin{array}{ccccccccc} \mathcal{C}(\mathbb{1}) & \subset & \mathcal{C}(x) & \subset & \mathcal{C}(\bar{x}\otimes x) & \subset & \mathcal{C}(x\otimes\bar{x}\otimes x) & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathcal{C}(\mathbb{1}) & \subset & \mathcal{C}(\bar{x}) & \subset & \mathcal{C}(x\otimes\bar{x}) & \subset & \cdots \end{array}$$

$\mathcal{C} \overset{F}{\curvearrowright} A$: An action of a UTC $\mathcal{C}$ on a $*$-Algebra A is a full and faithfull tensor functor

bi-involutive functor $\underbrace{F : \mathcal{C} \overset{\otimes}{\hookrightarrow} \mathrm{Bim}(A).}_{\text{"}}$

Eg: $\Gamma \overset{\alpha}{\curvearrowright} A$

$\alpha : \mathrm{Hilb}(\Gamma) \overset{\otimes}{\hookrightarrow} \mathrm{Bim}(A)$

UTC-actions on C^*-algebras:

Do UTCs act on C^*-algebras?

Yes! Izumi: [Izu93] constructed certain UTC-actions on Cuntz algebras using type III-factors and ideas from QFT:

$\mathbb{1}, \tau$ Fib $\curvearrowright \mathcal{O}_2$, $\mathrm{Rep}(S_3) \curvearrowright \mathcal{O}_3$.

Yuan [Yu19] constructed UTC-actions on non-separable C^*-algebras.

von Neumann algebras v.s. C^*-algebras

By [PopShl93], $L\mathbb{F}_\infty$ is a universal receptacle for UTC-actions.

By K-theoretic obstructions, $\nexists$ universal monotracial separable C^*-alg admitting actions by every UTC.

Theorem: [Hartglass, HP 20] Every UTC acts on some simple separable unital monotracial GJS C^*-algebra B_0.

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This is given $\mathcal{C}$,

$\exists\, F : \mathcal{C} \overset{\otimes}{\hookrightarrow} \mathrm{Bim}^{tr}_{fgp}(B_0)$.

Pf sketch:

Fix a generator $x \in \mathcal{C}$.

1 A_∞ AF C^*-algebra from

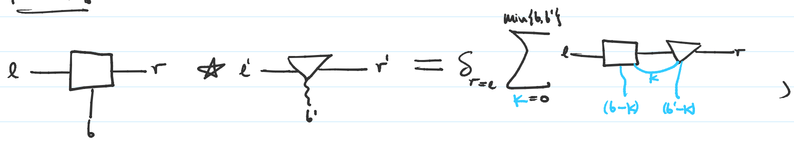
$\cdots \subset \mathcal{C}(x^{\otimes n}) \subset \mathcal{C}(x^{\otimes(n+1)}) \subset \cdots$

① A_∞ AF C^*-algebra from

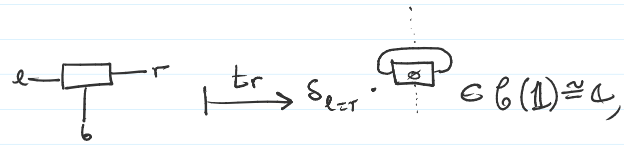
$$\cdots \subset \mathcal{C}(x^{\otimes n}) \subset \mathcal{C}(x^{\otimes(n+1)}) \subset \cdots$$

$A_\infty - A_\infty$ Bimodule: $\underbrace{Gr_\infty := \bigoplus_{b,l,r \geq 0} \mathcal{C}(x^{\otimes b} \to x^{\otimes l} \otimes x^{\otimes r})}_{\text{Graded tracial } *\text{-algebras}}$

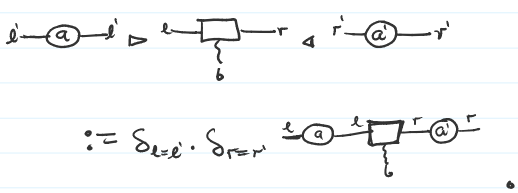
Product:



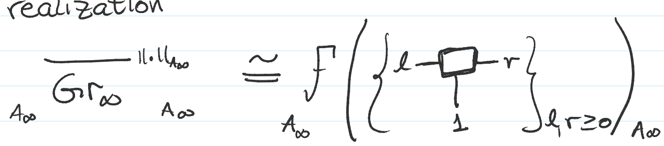
Tr:



$A_\infty - A_\infty$ actions:

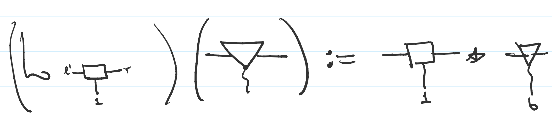


② Completing in A_∞-norm, get full Fock Space realization



$$= A_\infty \oplus \bigoplus_{n \geq 1} X_1^{\boxtimes_{A_\infty} n}.$$

Providing us with creation/annihilation operators



③ Using GNS with Tr:

$$B_\infty := C^*\left(A_\infty, \{L_\xi\}_{\xi = \xi^*}\right) \subseteq \mathcal{B}^\dagger\left(\overline{Gr_\infty}_{A_\infty}\right)$$

$$\underbrace{B_\infty}_{A_\infty\text{-valued semicircular system!}} := C^*\left(A_\infty, \{L_\xi\}_{\xi = \xi^*}\right) \subseteq \mathcal{B}^\dagger\left(\overline{Gr_\infty}_{A_\infty}\right)$$

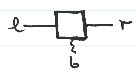
Together with

$$\underbrace{B_\infty \xrightarrow{E} A_\infty \xrightarrow{Tr} \mathbb{C}}_{\text{semifinite tracial weight}} .$$

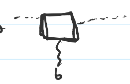
Corners in B_∞

$\forall n$: projection $P_n := n$ —— $n = P_n^* = P_n^2$,

$${}_\ell B_r := P_\ell \cdot B_\infty \cdot P_r \ni$$



$$B_0 := {}_0B_0 := \emptyset \cdot B_\infty \cdot \emptyset \ni$$



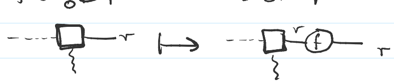
Using diagrams:

$${}_{B_0}\left({}_0B_n \underset{B_0}{\boxtimes} {}_0B_m\right)_{B_0} \cong {}_{B_0}\left({}_0B_{(n+m)}\right)_{B_0}$$

4 $\quad F : \mathcal{C} \xhookrightarrow{\otimes} \mathrm{Bim}_{fgp}(B_0)$

$$X^{\otimes r} \mapsto {}_0B_r$$

$$\underbrace{r — (f) — r'}_{\mathcal{C}(X^{\otimes r} \to X^{\otimes r'})} \mapsto \quad Ff : {}_0B_r \longrightarrow {}_0B_{r'}$$



By construction, F is faithful $\otimes$-functor.

Showing F is full requires some analysis!

5 $\quad$ Hilbertification of our C^*-bimodules

Idea: Turn Banach Space into Hilbert Space:

$$\left(— \underset{B_0}{\boxtimes} L^2(B_0)\right)$$

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "{[BHP13]}"] \arrow[d, phantom, "\cong \Rightarrow"] & \\
\mathrm{Bim}^{tr}_{fgp}(B_0) \arrow[rr, hook] & & \mathrm{Bim}^{sp}_{bf}(M_0)
\end{tikzcd}$$

full using vN algs

$${}_{B_0}\mathcal{Y}_{B_0} \longmapsto {}_{M_0}\left(\mathcal{Y} \boxtimes_{B_0} L^2 B_0\right)_{M_0}$$

non-trivial extensions

+ Finite-dimensional linear algebra completes the proof. ■

(weaker) universality:

Corollary: Every Unitary Fusion Category (UFC) acts on the same C^*-algebra

Pf: $\ast$[UFC] $\cong$ UTC ■

Perspectives: Classification of 'amenable' C^*-algebras was recently achieved at a similar level of Connes' classification of injective factors [Con76].

C^*:	vN:
'Classifiable' C^*-algebras are $\mathcal{Z}$-stable: $A \otimes \mathcal{Z} \cong A$	Injective inf-dimensional factors are R-stable: $M \otimes R \cong M$
The Jiang-Su algebra $\mathcal{Z}$ is an inf-dim analogue of $\mathbb{C}$, and is the minimal self-absorbing C^*-alg.	

By K-theoretic obstructions, only integral UFCs can act on $\mathcal{Z}$ ($\mathrm{Vec}(G,\omega)$, $\mathrm{Rep}(G)$, ...)

: $\mathcal{Z} \longleftrightarrow R$: admits a unique action from every UFC.

[EGW]: $\mathrm{Vec}(G,\omega) \curvearrowright \mathcal{Z}$!
$\Rightarrow \omega = 1$

None $\longleftrightarrow L\mathbb{F}_\infty$: admits an action from every UTC

What are the quantum symmetries of $\mathcal{Z}$?

Does $\mathrm{Rep}(S_3) \curvearrowright \mathcal{Z}$?

Are C^*-subalgebras characterized by a standard invariant?

Thank You!

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